On the Problem of Closure Conditions in Particular for Vlasov Turbulence

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In turbulence-theory, the equations for the dynamics of statistical quantities usually form an infinite system. The coupled momentum equations are the best known example. Such relations can also be derived for the n-point-distribution functions which describe the stochastic state of a turbulent medium at every instant. This will be shown for the case of Vlasov turbulence. In order to obtatin a solution, it is a common procedure, to terminate such infinite systems of equations by closure-conditions. Often this is considered as an approximation. In the case of Vlasov-turbulence it can be shown that a wide variety of closure conditions, all of which are exactly admissible, exists.

Concerning recent turbulence theories, KRAICHNAN has considered the coincidence of theoretical forecasts and experimental evidence as an "accident or a miracle" [1]. The situation may be even worse, since this statement refers mainly to the difficulties in obtaining approximate solutions for the statistical equations of motion. It leaves out of consideration the fact that the usual assumption of an initial Gaussian distribution is only one of a wide variety of possibilities, in chosing the initial statistics of the turbulent field. Actually one can only hope that the results become insensitive for larger times to a variation of the special selected initial conditions.

In some sense it is possible to define exact closure conditions in order to cut the infinite system of moment equations. In fact there should exist a wide variety of such possibilities, which corresponds to the uncertainty in the choice of the initial statistics.

Our discussion is mainly based on a hierarchy of equations of the time variation of the joint probabilities which describe the instantanious turbulent state of the fluid from a stochastic point of view. They correspond very closely to the wellknown BBGKY-hierarchy and express the time-variation of the n-point distribution by means of suitably selected $(n+1)$-point distributions. Relations of this type have been derived, presumably for the first time, in the case of hydrodynamical turbulence by LUNDGREN [2]. Using a somewhat different formal procedure, which is suggested by Bernsteins justification of the Fokker-Planck-Kolmogorov-equation [3] we shall give in the first chapter a derivation for our case of interest. The resulting hierarchy of equations may be confirmed by a somewhat more direct physical inspection.

The initial value problem is discussed in the next chapter. All possible initial conditions can be separated into classes, which are equivalent in the sense that they fix the history of the 1- or 2-point distribution functions. On the other hand there is a lack of a good philosophy which would enable one to select a best initial condition in agreement with any special given experimental situation.

In the 3rd chapter, we shall treat the hierarchy equations as recurrence relations for the distribution functions of growing order. This allows us to define the "admissibility" of exact closure conditions. Corresponding to the uncertainty in the initial conditions, one may expect a wide variety of possibilities for cutting the infinite set of equations.

In the last chapter a simple case of such recurrence relations is treated explicitly. We discuss the restrictions imposed on the 2-point distribution, if the 1-point distribution is asumed to be known for all times.

1. The Lundgren Hierarchy for Vlasov Turbulence

We write Vlasov's equation in units $e=m=1$ with fixed ion background in the form

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + E\frac{\partial f}{\partial u} = 0\,, \tag{1 a}$$

$$\frac{\partial E}{\partial x} = \int f \,\mathrm{d}u - 1\,. \tag{1 b}$$

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[1] R. KRAICHNAN, in: Pai, Dynamics of Fluids and Plasmas, Academic Press New York 1966, p. 239.

[2] T. S. LUNDGREN, Phys. Fluids **10**, 1969 [1967].

[3] Compare P. LEVY, Processus Stochastiques et Mouvement Brownien. 2nd ed., Gauthier-Villars, Paris 1965, no. 16.

Also we consider only the 1-dimensional case and shall demand as usual, over all-neutrality

$$I = \frac{1}{2L} \int_{-L}^{+L} dx \int du\, f(x,u) \to 1 \quad \text{for} \quad L \to \infty .$$

Whereas f is commonly treated as a probability-density for particles, we shall forget this aspect in favour of some kind of a fluid-interpretation. This enables us to set up a statistical description for f with which we can treat turbulence theory for Vlasovs equation. Then f becomes a stochastic function of x and u for every fixed time.

But this implies that also the integral I becomes a stochastic variable. Hence, over all neutrality is not expected to hold automatically, if appropriate "selection rules" are not observed by the statistics itself. This must be assumed only initially if we assume

$$\lim_{|u| \to \infty} \text{Prob}\{u : f > 0\} = 0$$

since then we expect I to become constant in time.

A practically important special case covers this relation; that is, if the correlation length of $\delta f = f - \bar{f}$ is finite:

$$\infty > c \geqq \int\int \langle \delta f(x,u)\, \delta f(y,v) \rangle \, dy \, dv .$$

In fact, if

$$\bar{I} = \frac{1}{2L} \int_{-L}^{+L} dx \int du\, \bar{f}(x,u)$$

is the mean value, we get for the fluctuation

$$\langle (I - \bar{I})^2 \rangle = \frac{1}{4L^2} \int \dots \int dx\, du\, dy\, dv \cdot \langle \delta f(x,u)\, \delta f(y,v) \rangle \leqq c/2L$$

which tends to zero in the limit $L \to \infty$. Hence the distribution of I is sharp around 1:

$$\text{Prob}(I) = \delta(I-1) .$$

The situation is more difficult if we work with finite boundary conditions. Then it may be convenient to change Poisson's equation into

$$\frac{\partial E}{\partial x} = \int f\, du - \frac{1}{L} \int_0^L dx \int f\, du$$

and thereby automatically adjust the ion-background in a suitable manner.

In the following, we shall frequently treat the stellardynamic case for simplicity. This is defined by the lack of a neutralizing background, according to the fact that the gravitating forces are only attracting. Then Poisson's equation reads simply

$$\frac{\partial E}{\partial x} = \int f\, du . \tag{2}$$

Since it is evident from the formulas, we shall not always explicitly state which case is under treatment in the following.

Lundgren for the first time has shown that the concept of the BBGKY-hierarchy can also be carried over to the case of fluid media with a continuum of degrees of freedom, for instance those described by Navier-Stokes equations. Instead of carrying over his method from this case to our Vlasov-fluid we shall use a somewhat different procedure which was essentially given by the mathematician Bernstein in his derivation of the Fokker-Planck-equations [3].

Assume φ as an arbitrary function in one argument and define

$$M(t,x,u) := \mathcal{E}\{\varphi(f(t,x,u))\}$$

where $\mathcal{E}$ means "expectation". (Here we use this symbol to make the operational character of averaging more evident.) The time variation of M can be expressed in a twofold way: Since f as a stochastic field is characterized by its joint-distribution functions,

$$P_1(t,x,u:f)\, df = \text{Prob}\{f \leqq f(t,x,u) < f + df\}$$

$$P_2(t,x,u:f;x',u':f')\, df\, df' = \text{Prob}\{f \leqq f(t,x,u) < f + df \text{ and } f' \leqq f(t,x',u') < f' + df'\}$$

and so on, we have immediately

$$\frac{\partial M}{\partial t} = \int \frac{\partial P_1(t,x,u;f)}{\partial t} \varphi(f)\, df \tag{3}$$

by definition of expectation-values. On the other hand, using the equation of motion, we get independently

$$\frac{\partial M}{\partial t} = \mathcal{E}\left\{\frac{d\varphi}{df}\frac{\partial f}{\partial t}\right\} = -\mathcal{E}\left\{\frac{d\varphi}{df}\left(u\frac{\partial f}{\partial x} + \int G(x-x')\frac{\partial}{\partial u} f(x',u')\, f(x,u)\, dx'\, du'\right)\right\}.$$

Here G is a Greens' function which solves for the electric field by means of Poissons equation (2). If we interchange the linear operations, differentiation, and integration with $\mathcal{E}$,

$$\frac{\partial M}{\partial t} = -u\frac{\partial}{\partial x}\mathcal{E}\{\varphi(f(t,x,u))\} - \int\int G(x-x')\frac{\partial}{\partial u}\mathcal{E}\{f(t,x',u')\,\varphi(f(t,x,u))\}\, dx'\, du'$$

or explicitly, by definition of $\mathcal{E}$,

$$= -\int u \frac{\partial P_1(t,x,u:f)}{\partial x} \varphi(f)\, df - \iint dx'\, du'\, G(x-x') \frac{\partial}{\partial u} \iint P_2(t,x,u:f;x',u':f')\, f'\, \varphi(f)\, df\, df'.$$

This can be compared with (3). Because of the arbitrariness of $\varphi(f)$ we conclude:

$$\frac{\partial P_1}{\partial t} + u \frac{\partial P_1}{\partial x} + \iint dx'\, du'\, G(x-x') \frac{\partial}{\partial u} \int P_2(t,x,u:f;x',u':f')\, f'\, df' = 0. \tag{4}$$

This is the first of the hierarchy-equations. Using a function φ_2 of two arguments and with it:

$$M_2 := \mathcal{E}\{\varphi(f(t,x,u),\, f(t,x',u'))\}$$

one finds by a quite parallel procedure a corresponding relation for the time-variation of P_2,

$$\frac{\partial P_2}{\partial t} + u \frac{\partial P_2}{\partial x} + u' \frac{\partial P_2}{\partial x'} + \iint dx''\, du''\, G(x-x'') \frac{\partial}{\partial u} \int P_3(t,x,u:f;x',u':f';\; x'',u'':f'')\, f''\, df''$$
$$+ \iint dx''\, du''\, G(x'-x'') \frac{\partial}{\partial u'} \int P_3(t,x,u:f;x',u':f';\; x'',u'':f'')\, f''\, df'' = 0.$$

In a similar way the higher equations of the hierarchy can be obtained.

The above equations admit another interpretation if we introduce conditional probabilities. Call

$$\mathfrak{p}_1(t;x',u':f' \mid x,u:f) = P_2(t,x,u:f;x',u';f')/P_1(t,x,u:f).$$

Then we may rewrite (4) in the following form:

$$\left\{\frac{\partial P_1}{\partial t} + u \frac{\partial P_1}{\partial x} + \frac{\partial}{\partial u} P_1(t,x,u:f) \cdot \int G(x-x')\, \mathfrak{p}_1(t,x',u':f' \mid x,u:f)\, f'\, df'\, dx'\, du'\right\} = 0. \tag{5}$$

If we define a conditional electric field by the two equations:

$$E(t,x \mid x_1,u_1,f_1) = \int G(x-x')\, f(t,x',u')\, dx'\, du',$$
$$f(t,x=x_1,u=u_1) = f_1$$

then evidently we have within (5) an averaged electric field

$$\bar{E}(t,x,u;f) = \bar{E}(t,x \mid x,u,f) \tag{6}$$

where the average is taken over all the realisations of f which are consistent with the condition that the function f should have the value f at (x,u). With this definition the equation for P_1 takes the following form

$$\frac{\partial P_1}{\partial t} + u \frac{\partial P_1}{\partial x} + \frac{\partial}{\partial u} \bar{E}(t,x,u;f)\, P_1 = 0. \tag{7}$$

This equation can be confirmed by a somewhat more direct derivation. We get, using Vlasovs equation

$$\dot{M}_1 = -\mathcal{E}\left\{\frac{d\varphi}{df} u \frac{\partial f}{\partial x}\right\} - \mathcal{E}\left\{\frac{d\varphi}{df} E(t,x) \frac{\partial f}{\partial u}\right\}.$$

The first term on the right hand side has been treated earlier. For the second we have

$$-\mathcal{E}\left\{E(t,x)\frac{\partial \varphi}{\partial u}\right\} = -\frac{\partial}{\partial u}\mathcal{E}\{E(t,x)\,\varphi\}$$

or by definition of the $\mathcal{E}$-symbol

$$-\frac{\partial}{\partial u}\int \mathrm{Prob}\{t,x,u:f,E\}\, \varphi(f)\, E\, df\, dE.$$

Define the conditional probability:

$$\frac{\mathrm{Prob}\{t,x,u:f,E\}}{P_1(t,x,u:f)} =: \mathfrak{p}(t,x,u,f:E)$$

we have

$$\bar{E}(t,x \mid x,u,f) = \int \mathfrak{p}(t,x,u,f:E)\, E\, dE$$

where the field is that of (6). Hence for P_1 we get again (7).

These results can be checked further by a somewhat more pictorial description which again leads to another derivation of the Lundgren hierarchy. To this end we remark that P_1 describes the percentage of all realisations $f(t,x,u)$, for which

$$f_1 \leqq f(t,x_1,u_1) < f_1 + df.$$

Hence in deriving an equation for the time dependence of P_1, we should consider only those realisations $f(t,x,u)$ which are restricted by the above relation. We then also expect only a restricted set of possible fields $E(t,x \mid x_1,u_1,f_1)$ defined by

$$\frac{\partial E}{\partial x} = \int f(t,x,u)\, du - 1$$

and

$$f(t,x_1,u_1) = f_1.$$

Correspondingly, there is some well defined probability that these restricted electrical fields have a value within a given interval $(E, E+\mathrm{d}E)$ at point (x,t), say

$$\mathfrak{p}(t,x\colon E \,|\, x_1, u_1, f_1)\,\mathrm{d}E\,.$$

If there were only one realisation for E which means if E were sharply distributed — that this actually is possible will be shown later[4] — then we expect $P_1(t, x_1, u_1\colon f_1)$ to develop according to the characteristic mapping:

$$P_1(t+\mathrm{d}t, x_1, u_1\colon f_1) = P_1(t, x_1 - u_1\,\mathrm{d}t, u_1 - E_1\,\mathrm{d}t) \tag{8}$$

since the value of the function f remains unchanged along any characteristic path.

In the general case we expect an average over the different electric fields. In order to have at time $t+\mathrm{d}t$ the relation

$$f(t+\mathrm{d}t, x_1, u_1) = f_1$$

we must select the following sample functions at time t: If at time t the electric field has a value E, then $f(x,t,u)$ has to be restricted by

$$f(t, x_1 - u_1\,\mathrm{d}t, u_1 - E_1\,\mathrm{d}t) = f_1\,.$$

The corresponding probability is by definition

$$\mathfrak{p}(t, x_1 - u_1\,\mathrm{d}t, E \,|\, x_1 - u_1\,\mathrm{d}t, u_1 - E\,\mathrm{d}t, f_1)\,.$$

Hence we get by superposition:

$$\begin{aligned} P_1(t+\mathrm{d}t, x_1, u_1\colon f_1) &= \int \mathrm{d}E\,\mathfrak{p}(t, x_1 - u_1\,\mathrm{d}t, E \,|\, x_1 - u_1\,\mathrm{d}t,\ u_1 - E\,\mathrm{d}t\colon f_1) \\ &\quad \cdot P_1(t, x_1 - u_1\,\mathrm{d}t, u_1 - E\,\mathrm{d}t\colon f)\,. \end{aligned}$$

Expanding yields

$$\frac{\partial P_1}{\partial t} = -\int \mathrm{d}E\,\mathrm{d}u_1 \frac{\partial \mathfrak{p} P_1}{\partial x_1} - \int \mathrm{d}E \frac{\partial \mathfrak{p} P_1}{\partial u_1} E$$

which due to the normalization gives

$$\frac{\partial P_1}{\partial t} + u_1 \frac{\partial P_1}{\partial x_1} + \frac{\partial}{\partial u_1} \bar{E} P_1 = 0$$

with the former meaning of $\bar{E}$.

The characteristic equations of P_1 are the following:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u\,,$$
$$\frac{\mathrm{d}u}{\mathrm{d}t} = \bar{E}(t, x \,|\, x, u, f)\,,$$
$$\frac{\mathrm{d}P}{\mathrm{d}t} = -\frac{\partial \bar{E}}{\partial u}(t, x \,|\, x, u, f)\,.$$

[4] See the following paper: P. GRÄFF, Z. Naturforsch. **24 a**, 711 [1969].

Hence we have if the solutions of $\dot{x}, \dot{u}$ are

$$x(\tau \,|\, t_1, x_1, u_1)\,; \qquad u(\tau \,|\, t_1, x_1, u_1)$$

$$\begin{aligned} P_1(t_1, x_1, u_1\colon f_1) &= P_1(0, x(0 \,|\, t_1, x_1, u_1), u(0 \,|\, t_1, x_1, u_1)) \\ &\quad - \int_0^{t_1} \frac{\partial}{\partial u} \bar{E}(x(\tau \,|\, t_1, x_1, u_1), u(\tau \,|\, t_1, x_1, u_1))\,\mathrm{d}\tau\,. \end{aligned}$$

Now we shall explain a "mixing rule" which allows us to extend the given results to a wider class of solutions. Here we use the fact that Lundgrens equations admit convex linear superpositions. Actually this is to be expected since they describe the statistical behaviour of ensembles.

In particular, let

$$\{P_1', P_2', \ldots\} \quad \text{and} \quad \{P_1'', P_2'', \ldots\}$$

be two solutions of the hierarchy. Then the following convex linear-combination is again a solution:

$$\begin{aligned} P_1 &= \cos^2\alpha \cdot P_1' + \sin^2\alpha \cdot P_1''\,, \\ P_2 &= \cos^2\alpha \cdot P_2' + \sin^2\alpha \cdot P_2''\,, \\ &\vdots \end{aligned}$$

For a proof one has to show that this convex combination is again positiv, symmetric with respect to any permutation, e. g.

$$P_2(t, x_1, u_1\colon f_1; x_2, u_2\colon f_2) = P_2(t, x_2, u_2\colon f_2; x_1, u_1\colon f_1)\,, \tag{9}$$

and obeys Lundgrens equations if this is the case for P', P''. This is an immediate consequence of the linearity and homogeneity.

Of course this result can be extended to the case of finite and infinite linear-combinations:

Let $\qquad \mathfrak{p}_i \geqq 0 \quad \text{and} \quad \sum \mathfrak{p}_i = 1\,.$

Then, if

$$\{P_1^{(i)}, P_2^{(i)}, \ldots\} \qquad i = 1, 2, \ldots$$

is a sequence of solutions, the same holds for

$$P_k = \sum \mathfrak{p}_i P_k^{(i)}.$$

The index-set of i need not be discrete. A simple example is the following: Let

$$P = \{P_1, P_2, \ldots\}$$

be a solution for the unbounded x-space. Then

$$\begin{aligned} P_1^{\xi} &:= P_1\ (t, x+\xi, u\colon f)\,, \\ P_2^{\xi} &:= P_2\ (t, x+\xi, u\colon f; x'+\xi, u'\colon f')\,, \\ &\vdots \end{aligned}$$

is again a solution for any value of ξ. And due to our superposition-theorem the same holds for

$$\bar{P}_1 = \frac{1}{2L}\int_{-L}^{+L} d\xi \, P_1^{\xi},$$
$$\bar{P}_2 = \frac{1}{2L}\int_{-L}^{+L} d\xi \, P_2^{\xi}, \qquad (10)$$
$$\vdots$$

which represent in some sense smeared-out distributions.

2. The Initial-Value Problem for the Statistics

Evidently, the derived hierarchy of equations corresponds to Liouville's equation in ordinary mechanics of point particles. In fact, they determine the statistical information on the stochastic field $f(x, u)$ at time $t > 0$ by its original statistics at time $t = 0$. Suppose the latter are given,

$$P_1^0(x, u: f) := P_1(t=0, x, u: f),$$
$$P_2^0(x, u: f; x', u': f') := P_2(t=0, x, u: f; x', u': f'),$$
$$\vdots$$

then the Lundgren-equations allow us to calculate P_n for all $t \neq 0$ at least in principle.

Actually we shall be interested only in some of the lowest order distribution-functions. For instance, if we were interested in $P_1(t, x, u: f)$, we would not need the full probability-measure at time zero, but only some very special moments of the higher distributions. This can be seen as follows:

We have by (4)

$$\frac{\partial P_1}{\partial t} = -u\,\frac{\partial P_1}{\partial x} - \int G(x-x') \cdot \frac{\partial Q_1}{\partial u}(t, x, u, f; x', u')\, dx'\, du'$$

if we abbreviate

$$Q_k(t, x, u: f; x_1, u_1, \dots x_k, u_k) = \int P_{k+1}(t, x, u: f; x_1, u_1: f_1; \dots x_k, u_k: f_k) \prod_{j=1}^{k} f_j \, df_j \,.$$

But for the Q_k we get the following hierarchy of equations which itself is closed inasfar as it contains only "Q-functions":

$$\frac{\partial Q_k}{\partial t} + u\,\frac{\partial Q_k}{\partial x} + \iint G(x-x_{k+1})\,\frac{\partial}{\partial u} Q_{k+1}\, dx_{k+1}\, du_{k+1} + \sum_{j=1}^{k}\left\{u_j\,\frac{\partial Q_k}{\partial x_j} + \iint G(x_j - x_{k+1})\,\frac{\partial}{\partial u_j} \cdot Q_{k+1}\, dx_{k+1}\, du_{k+1}\right\} = 0\,.$$

Hence it is obvious, that twe do not need the full information about the initial measure, but only the sequence

$$\{P_1^0, Q_1^0, Q_2^0, \dots\} \qquad (11)$$

in order to calculate the whole history of P_1. In this respect all initial measures

$$P^0 = \{P_1^0, P_2^0, \dots\}$$

with the same sequence (11) are equivalent.

Besides $P_1(t=0)$ one may be tempted to prescribe only the Q's in an arbitrary way initially. But it should be kept in mind, that the Q's demand a representation as moments of distribution functions, which could themselves serve as a description of a possible initial measure. Formally this means the Kolmogorov requirements hold:

$$\int P_k \, df_k = P_{k-1}\,,$$
$$P_k \text{ symmetric (formula 9)}.$$

Only those Q's are admisible, for which at least one equivalent measure exists. Similar to KRAICHNAN [5], one may call this a realizability-condition.

Now we are at a stage, in which it seems useful to clarify the "advantage or not", brought up by a statistical treatment of turbulence.

(i) In a first step we introduced the Vlasov-distribution in order to take into account the uncertain knowledge of positions and velocities of the individual particles. This was a first probabilization.

(ii) Since Vlasovs equation was able to describe collective effects such a plasma waves, a second probabilization seemed useful, especially if one wants to deal with the statistical turbulence of these waves.

But we have to guess something about $P^0(f)$ to make turbulence theory a definite problem. This has been called the gap-problem [6]. In ordinary Navier-Stokes-turbulence one commonly introduces a Gaussian measure at $t = 0$, whereas in our case some kind of "random phases" seems to be preferable. Comparing different initial measures, one could think also of a third probabilization – the counterpart of a so called Bayes'ian strategy.

Another posibility would be to treat all initial measures of a suitable class π_0 on an equal footing and to ask for the "best value" of some parameter H. For instance

$$H(t) := \inf_{P^0 \in \pi^0} \int P_1(t, x, u: f) \ln P_1(t, x, u: f)\, df\, dx\, du$$

[5] R. H. KRAICHNAN, J. Math. Phys. 2, 124 [1961].
[6] P. GRÄFF, MPI-PAE/PL 5/68 (März 1968).

with

$\pi_0 := \{P^0\colon \int P_1(t=0) \ln P_1(t=0) \, df \, dx \, du = H_0\}$.

This last type is connected with the socalled minimax-strategies. (For a more detailed description of these methods compare WITTING [7] or WALD [8].)

Hence, given initially only, say, the first distribution function P_1 $(t=0)$, we expect no unique development in the course of time. Rather it depends upon the way, in which we complete P_1 $(t=0)$ by the Q's initially. According to different choices, we get a "diffusion" for P_1 $(t=0)$. This can be described very nicely from the Bayesian point of view since then we could calculate an average

$$\langle P_1(t>0, x, u\colon f) \rangle$$

and a fluctuation around it

$$\langle \{P_1(t>0) - \langle P_1(t>0) \rangle \}^2 \rangle.$$

All these considerations are nearly trivial and straight-forward. They show the difficulty in speaking of "the solution of the turbulence-problem". Evidently a further argument is desired, perhaps something like "coarse-graining".

This will be confirmed, as we discuss in the following the exact closure conditions.

3. Closure-Conditions

Another possibility of achieving a uniquely defined version of the turbulence problem involves closure conditions. Commonly these are introduced in order to terminate an infinite system of equations. The results, it is hoped, approximate in a suitable way the — or at least one — solution of the infinite set or, more correctly, its projection onto the subspace of interest. Due to the earlier considerations we would be doubtful about the usefulness of a concept such as "approximation".

To be more explicit, multiply (4) by f and integrate:

$$\frac{\partial \overline{f(t,x,u)}}{\partial t} + u \frac{\partial \overline{f}}{\partial x} + \iint dx' \, du' \, G(x-x') \frac{\partial}{\partial u} \overline{f(t,x,u) \, f(t,x',u')} = 0. \quad (12)$$

Of course we could have gained this easier, using only the linear character of averaging. The simplest closure-ansatz would tempt one to express the correlation by the mean values:

$$\overline{f(t,x,u) \, f(t,x',u')} = \text{functional}\,(t,x,u,x',u'; \overline{f(t)}).$$

Substituting into (12), we find as the solution of this equation a well determined function $g_{f^0}(t,x,u,)$ for every initial value f^0 of f. As already remarked, g is sometimes considered as an approximation for $\overline{f}$ for $t \neq 0$. But instead we may ask, whether there exists — at least one — initial measure P^0 and as a consequence a measure for all times, in such a way, that it reproduces just $g_{f^0}(t,x,u)$:

$$g_{\overline{f}^0}(t,x,u) = \int P_1(t,x,u\colon f) \, f \, df.$$

If this were the case, we would be certain at least formally that we were in accordance with an exact solution of the turbulence problem.

In a similar way we may ask for a closure of Lundgren's equations directly, for instance (4). This would mean requiring the $Q_1(t,x,u\colon f; x' u')$ for all times as a function of P_1:

$$Q_1(t,x,u\colon f; x',u') = \Psi(t,x,u\colon f; x',u'; \{P_1\}).$$

Together with an initial condition for P_1:

$$P_1(t,x,\mathrm{u}\colon f)|_{t=0} =: P_1{}^0(x,u\colon f) \quad (13)$$

this defines a function $\Gamma(t,x,u\colon f)$ for which we can again ask whether it can be interpreted as a possible P_1-distribution or not. This corresponds to the question of whether there exists an initial measure P^0 with $P_1{}^0$ identical to that given by (13) which reproduces just Γ.

These considerations lead to the following definition: We call the combination $\{\Psi, P_1{}^0\}$ admissible, if the resulting Γ can be interpreted as a possible P_1. We call Ψ a closure condition which is uniformly admissible on $\mathfrak{P}$, if al the combinations $\{\Psi, P_1{}^0\}$ with $P_1{}^0 \in \mathfrak{P}$ are admissible.

As is well known, Millionstchikovs „Quasi-Gaussian" gives a lot of good results. But it is surely not uniformly admissible for all initial conditions, yielding sometimes negative energy-spectra. In a similar way one may set up definitions for the higher distributions of the hierarchy. Since any of the above mentioned combinations fixes one function (Γ), we can speak immediately of the admissibility of these directly. Then, if P_2 is admissible, the same

[7] H. WITTING, Mathematische Statistik, Verlag Teubner, Stuttgart 1966, esp. chapt. 1.5.

[8] A. WALD, Statistical Decision Functions, Wiley & Sons, New York 1950.

holds of course for the correponding P_1

$$P_1(t,x,u:f) = \int P_2(t,x,u:f;x',u':f')\,\mathrm{d}f'.$$

From this point of view, the main task is to find out, whether a given P_k is admissible or not. The equations of interest are:

$$\frac{\partial P_k}{\partial t} + \sum_{j=1}^{k} u_j \frac{\partial P_k}{\partial x_j} + \sum_{j=1}^{k} \iint \mathrm{d}x_{k+1}\,\mathrm{d}u_{k+1}\,G(x_j - x_{k+1}) \frac{\partial}{\partial u_j} \cdot \int P_{k+1} f_{k+1}\,\mathrm{d}f_{k+1} = 0\,, \tag{14 a}$$

$$P_k - \int P_{k+1}\,\mathrm{d}f_{k+1} = 0\,. \tag{14 b}$$

Now the essential point is to consider them as recursion-formulas for the P_k's. Given P_n, one may ask whether (14 a) and (14 b) have – at least one – solution for P_{n+1} for which

$$P_{n+1} \geqq 0$$

and that further on a P_{n+2} exists in a similar manner and so on. It seems useful to introduce a weaker form of admissibility by the following definition:

P_n is admissible to first order, if at least one function P'_{n+1} exists, which is in accordance with the following relations:

$$\sum_{j=1}^{n} \iint \mathrm{d}x_{n+1}\,\mathrm{d}u_{n+1}\,G(x_j - x_{n+1}) \frac{\partial}{\partial u_j} \int P'_{n+1} f_{n+1}\,\mathrm{d}f_{n+1} = -\frac{\partial P_n}{\partial t} - \sum_{j=1}^{n} u_j \frac{\partial P_n}{\partial x_j}\,, \tag{15 a}$$

$$\int P'_{n+1}\,\mathrm{d}f_{n+1} = P_n\,, \tag{15 b}$$

$$P'_{n+1}\,(1, 2, \dots, n+1) = \text{symmetric} \tag{15 c}$$

where "i" abbreviates $(t, x_i, u_i : f_i)$ and

$$P_{k+1} \geqq 0\,. \tag{15 d}$$

Evidently, for P_k to be admissible it is necessary that it is admissible to first order.

In a similar way we may define P_n to be admissible to second order, if it is admissible to first order and its corresponding P'_{n+1} is itself again admissible to first order. Corresponding definitions can be established up to all orders. From these definitions the following relations can be concluded:

A) If P_n is admissible to first order, P_{n-1} is admissible to second order.

B) If P_n is admissible to any order, then also P_{n-1}.

C) If P_n is admissible, it is admissible to any order.

D) If P_n is admissible to any order it is admissible.

The last point follows from the remark that if P_n is admisisble to any order, a sequence P_k, $(k>n)$ can be constructed in accordance with the Kolmogorov requests, which hence defines a measure on f-space. This way a unique answer for the turbulence problem has been achieved, as far as the interesting sub-space is concerned.

4. Conditions on P_1

As a simple example we treat the admissibility of first order for P_1. This means that we assume P_1 to be given for all times. We have to ask for the existence of at least one $P_2 \geqq 0$ with

$$\int P_2(t,x,u:f;x',u':f')\,\mathrm{d}f' = P_1(t,x,u:f)\,, \tag{16 a}$$

$$\int P_2(t,x,u:f;x',u':f')\,f'\,\mathrm{d}f' = Q(t,x,u:f;x',u') \tag{16 b}$$

where Q is a solution of

$$\int G(x-x') \frac{\partial}{\partial u} Q(t,x,u:f;x',u')\,\mathrm{d}x'\,\mathrm{d}u' = -\frac{\partial P_1}{\partial t} - u\frac{\partial P_1}{\partial x}\,. \tag{17}$$

Besides this equation Q must obey some further restrictions which are necessary conditions in order to fulfill (15 c) and (15 d). If we remember that together with P_1 also

$$\bar{f}(t,x,u) := \int P_1(t,x,u:f)\,f\,\mathrm{d}f$$

is a known function, the following relations must hold:

$$\int Q(t,x,u:f;x',u')\,\mathrm{d}f = \bar{f}(t,x',u')\,, \tag{18 a}$$

$$\int Q(t,x,u:f;x',u')\,f\,\mathrm{d}f = \int Q(t,x',u':f';x,u)\,f'\,\mathrm{d}f'\,, \tag{18 b}$$

$$Q \geqq 0\,. \tag{18 c}$$

The last condition is a consequence of (15 d) and the fact that f as a probability should always be positiv:

$$\text{Prob}\,\{f<0\} = 0\,.$$

This way we have split the problem in accordance with our earlier remarks on the Q-hierarchy into two tasks:

First solve the problem for Q, and second that for P_2 which corresponds to controlling the "realizability" of Q.

In order to attack the first question, we think of Q as represented by some kind of expansion

$$Q(t,x,u:f,x',u') = \sum \chi_j(t,x',u')\,\pi_j(t,x,u:f). \tag{19 a}$$

Since we are not interested in the most general solution, let us assume

$$\chi_j \geqq 0; \qquad \pi_j \geqq 0 \tag{19 b}$$

which automatically ensures (18 c) to be fulfilled. Since by (18 a)

$$\int \pi_j \, \mathrm{d}f$$

should be independent of (x,u), we may normalize it to 1:

$$\int \pi_j(t,x,u:f)\,\mathrm{d}f = 1 \tag{20}$$

which shows π_j as a 1-point probability measure. Now equation (18 a) reads:

$$\sum \chi_j(t,x,u) = \bar{f}\,(t,x,u) \tag{21}$$

and if we define $\varepsilon_j(t,x)$ by

$$\frac{\partial \varepsilon_j(t,x)}{\partial x} = \int \chi_j(t,x,u)\,\mathrm{d}u$$

we arrive for (17) at

$$\sum_j \varepsilon_j(t,x)\,\frac{\partial \pi_j(t,x,u:f)}{\partial u} = -\frac{\partial P_1}{\partial t} - u\,\frac{\partial P_1}{\partial x}. \tag{22}$$

We are interested in those solutions of (22) which also obey (18 b), (19 b) and (20), the last of which, in view of our expansion (19 a) reads as:

$$\sum \chi_j(t,x',u')\,\bar{f}_j(t,x,u) = \sum \chi_j(t,x,u)\,\bar{f}_j(t,x',u') \tag{23}$$

if we designate by $\bar{f}_j$ the mean value with respect to π_j. The existence of such π_j would ensure that the Q-equation was solvable. The above relations may be further reformulated by the ansatz

$$\bar{f}_j(t,x,u) = \sum_k c_{jk}\,\chi_k(t,x,u). \tag{24}$$

Now, (23) yields immediately

$$c_{jk} = c_{kj}$$

and (19 b) postulates (perhaps with again some loss of generality)

$$c_{jk} \geqq 0.$$

But this is not the whole content of (19 b); let us take the moments of (22).

Zeroth order:

$$\sum \varepsilon_j \cdot \partial \pi_j^{(0)}/\partial u = 0$$

This can easily be fulfilled by

$$\pi_j^{(0)} = 1$$

as is demanded by (20) which hence does not contradict (22).

First order:

$$\sum_j \varepsilon_j\,\frac{\partial \bar{f}_j}{\partial u} = -\frac{\partial \bar{f}}{\partial t} - u\,\frac{\partial \bar{f}}{\partial x}.$$

Using (21) and (24) this can be written

$$\sum_j \left\{ \frac{\partial \chi_j}{\partial t} + u\,\frac{\partial \chi_j}{\partial x} + \sum_k \varepsilon_k\, c_{kj}\,\frac{\partial \chi_j}{\partial u} \right\} = 0.$$

Define E_j by

$$\frac{\partial E_j}{\partial x} = \sum_k c_{jk} \int \chi_k\,\mathrm{d}u. \tag{25}$$

We have as a sufficient relation for the χ's:

$$\frac{\partial \chi_j}{\partial t} + u\,\frac{\partial \chi_j}{\partial x} + E_j\,\frac{\partial \chi_j}{\partial u} = 0. \tag{26}$$

Hence we have the following restriction imposed by the Q-hierarchy onto the P_1's: Their mean-value $\bar{f}\,(t,x,u)$ should allow a representation as a sum (21) of functions $\chi_j \geqq 0$ which themselves follow a system of "Vlasov equations" of the form (25) and (26) with suitable coefficients c_{jk}.

One might call this result a Q-representation-theorem. In any case it is sufficient to ensure the existence of solutions for the Q-admissibility of P_1 to first order. To this end we have to show that if the above relations hold, we are in fact able to find positive π_j in accordance with (22). We consider the higher order moments of (22):

$$\sum_j \varepsilon_j\,\pi_j^{(2)} = -\int_{-\infty}^{u} \left(\frac{\partial \bar{f^2}}{\partial t} + u\,\frac{\partial \bar{f^2}}{\partial x} \right) \mathrm{d}u + q^{(2)}(t,x),$$

$$\vdots$$

$$\sum_j \varepsilon_j\,\pi_j^{(k)} = -\int_{-\infty}^{u} \left(\frac{\partial \bar{f^k}}{\partial t} + u\,\frac{\partial \bar{f^k}}{\partial x} \right) \mathrm{d}u + q^{(k)}(t,x).$$

$$\vdots$$

The q's on the right hand side are integration-constants which generally allow the corresponding $\pi_j(t,x,\mathrm{u}:f)$ to become positive. This shows, that no further serious restrictions besides (25) (26) are imposed by the Q-hierarchy in first order.

Up to now we have taken into account only those restrictions for P_1, which were brought up by the Q-representation theorem for $\bar{f}(t,x,u)$ and they seem to be rather weak. We would have gained a somewhat narrower class, if we also had postulated

c_{jk} to be diagonal, as we shall do later on [4], since then "admissibility" can be lifted through all orders.

Our next step to establish fully the first order admissibility of P_1 is defined by the Eqs. (15 c, d), (16) when we ask for the existence of a P_2. The main difficulty is introduced by the demand of positivity. If we neglect this for a moment, the following expression is a solution:

$$\begin{aligned} &P_2(t, x, u: f; x', u': f') \\ &= P_1(t, x, u: f)\, P_1(t, x', u': f') \\ &\quad - [\overline{f(t,x,u)\, f(t,x',u')} - \bar{f}(t,x,u)\, \bar{f}(t,x',u')]^{-1} \\ &\quad \cdot [Q(t,x,u: f; x', u') - P_1(t,x,u: f)\, \bar{f}(t,x',u')] \\ &\quad \cdot [Q(t,x',u': f'; x,u) - P_1(t,x',u': f')\, \bar{f}(t,x,u)] \\ &\quad + R(t,x,u: f; x',u': f') \qquad (27) \end{aligned}$$

(see the appendix). Here the correlation is determined by:

$$\overline{f(t,x,u)\, f(t,x',u')} = \int Q(t,x,u: f; x', u')\, f\, \mathrm{d}f$$

and R is such that

$$\int R\, \mathrm{d}f = \int R\, f\, \mathrm{d}f = 0\,, \qquad \int R\, \mathrm{d}f' = \int R\, f'\, \mathrm{d}f' = 0\,.$$

This is not the only representation for possible P_2's, as can be seen by means of the other example in the appendix.

Therefore, if there were not the demand for P_2 to be positiv, we had no further restriction on P_1 than the earlier mentioned Q-condition. The further restriction on P_1 which is brought up by positivity we may call the P-condition. In general it seems rather hard to find simple sufficient criteria for the P-condition to hold. A rather complicated possibility would be (27) $\geqq 0$.

Final Remarks

To make statistical turbulence theory a definite task, we would have to fix the initial probability distributions in a best way in accordance with the given experimental data. An equivalent possibility would correspond to the prescription of a well fitted admissible closure condition.

It seems difficult to bridge this gap by some kind of information theoretical procedure (as has been proposed in classical mechanics for instance by JAYNES [9]). Any such treatment presupposes an "a priory" measure as a kind of reference system, with respect to which any probability may be expressed in the form of a density. Such a measure is selected in classical mechanics by Liouville's theorem and ergodicity. However, the task of establishing the corresponding results for our situation remains.

We have considered the hierarchy of equations for the joint-probabilities as recursion formulas for the distribution functions of growing order. The essence of these considerations is to show that the finding of exact solutions of the statistical equations of motion is perhaps not the only problem. Actually in a forthcoming paper an exact treatment for the case of a product measure will be presented [4]. Even in this special case no unique version of the problem will be achieved. Due tot his lack, turbulence theory is to some extent "open".

Finally it is interesting to note that these difficulties correspond closely to the ones met in the quantum-theory of fields. Especially for the nonlinear spinor theory it has been shown by MITTER and ROOS [10] that a wide variety of possible sets for the vacuum expectation values are compatible with an arbitrary given 2-point Greens-function — which function coresponds to the correlation function in our case. Uniqueness may not be achieved before the influence of the commutation relations on the initial conditions is also taken into account.

I wish to thank Dr. D. PFIRSCH for valuable discussions.

Appendix

For any two fixed points (x, u) and (x', u'), denote:

$$F(f, f') = P_2(x, u: f; x', u': f')\,.$$

We ask for the existence for at least one $F \geqq 0$ if P_1 and Q_1 are known. If we call

$$\begin{aligned} a(f) &:= P_1(x, u: f)\,, \\ b(f') &:= P_1(x', u': f')\,, \\ A(f) &:= Q(x, u: f; x', u')\,, \\ B(f') &:= Q(x', u': f'; x, u) \end{aligned}$$

[9] E. T. JAYNES, Phys. Rev. **106**, 620 [1957].

[10] H. ROOS, Zur Struktur des Gleichungssystems für die Vakuumerwartungswerte einer nicht linearen Feldtheorie mit indefiniter Metrik, Dissertation, München 1966.

the only restrictions on F are the following:

$$\begin{aligned} \int F(f,f')\,\mathrm{d}f' &= a(f),\\ \int F(f,f')\,\mathrm{d}f &= b(f'),\\ \int F(f,f')\,f'\,\mathrm{d}f' &= A(f),\\ \int F(f,f')\,f\,\mathrm{d}f &= B(f'). \end{aligned}$$

For the existence of F, evidently the following conditions are necessary:

$$\begin{aligned} \bar a: &= \int a(f)\,\mathrm{d}f = 1,\\ \bar b: &= \int b(f')\,\mathrm{d}f' = 1,\\ \tilde a: &= \int a(f)\,f\,\mathrm{d}f = \int B(f')\,\mathrm{d}f' =: \bar B,\\ \tilde b: &= \int b(f')\,f'\,\mathrm{d}f' = \int A(f)\,\mathrm{d}f =: \bar A,\\ \tilde A: &= \int A(f)\,f\,\mathrm{d}f = \int B(f')\,f'\,\mathrm{d}f' =: \tilde B. \end{aligned}$$

Also, from the positivity of F it can be concluded that

$$\begin{aligned} \text{support }(A(f)) &\subset \text{support }(a(f)),\\ \text{support }(B(f')) &\subset \text{support }(b(f')). \end{aligned}$$

Also, for $b(y) \neq 0$, we have [11]

$$0 < \int_{-\infty}^{B(y)/b(y)} a(x)\,\mathrm{d}x < 1$$

and a similar relation for $a(x) \neq 0$.

Further we have

$$\int \mathrm{d}y\, b(y)\, \inf_x\, (A(x) - y\,\mathrm{d}a(x)) \leqq 0,$$

and

$$\int \mathrm{d}y\, b(y)\, \sup_x\, (A(x)\,y + a(x)) > \int (A^2 + a^2)\,\mathrm{d}x.$$

The question arises, whether these conditions also perhaps are sufficient. Surely this is the case, if we omit the positivity of F (and thereby the conditions). Then various solutions can be constructed. An example is the following:

$$F(f,f') = a(f)\,b(f') + \frac{1}{A - \tilde a\, b}\,(A - a\,\tilde b)\,(B - b\,\tilde a)$$

which we have used in the text. Another one may be gained as follows: Define the set of orthogonal Polynomials with respect to the "weight-function" $a(x)$ by $P_n(x)$ and with respect to $b(y)$ by $Q_n(y)$, and call

$$\begin{aligned} p(f) &:= P_1(f) / \int P_1(f)\,f\,\mathrm{d}f,\\ q(f) &:= Q_1(f) / \int Q_1(f)\,f\,\mathrm{d}f. \end{aligned}$$

Then, using the orthogonality of P, Q we prove

$$\begin{aligned} F(f,f') &= a(f)\,b(f')\\ &\quad + (A(f) - a(f)\,\tilde b)\,a(f')\\ &\quad + (B(f') - b(f')\,\tilde a)\,p(f)\\ &\quad - (\tilde A - \tilde a\,\tilde b)\,p(f)\,q(f') \end{aligned}$$

to solve the set of conditions.

All the above expressions are open up to an arbitrary additive function $R(f,f')$ whose moments of degree $\leqq 1$ all vanish. If we take arbitrary weight functions $\alpha(x)$, $\beta(y)$, we may achieve possible representations of such functions for instance by

$$R = \sum_{i \geqq 2} \sum_{k \geqq 2} P_i^{(\alpha)}(f)\, Q_k^{(\beta)}(f')\, \alpha(f)\, \beta(f')$$

if we denote the orthogonal functions in a similar way as before.

These considerations show, that the main restriction of physical interest must originate from the positivity of F (the P-restriction), which condition is hardly tractable due to its nonanalycity.

[11] I wish to thank Dr. R. WEGMANN for pointing out this fact to me.